The Chain Rule

We know how to differentiate:

$y = x^2 + 1$

But we don't know how to differentiate:

$y = (1 + x^2)^3$
$y = (1 + x^2)^{59}$
$y = \sqrt{1 + x^2}$
$y = sin(x^2 + 1)$

If the power is small enough, we could expand out the function and then take the derivative, but this gets time consuming for large powers. Instead we use the Chain Rule.

The Chain Rule - The Extended Power Rule

Examples:

Differentiate : $y = (1 + x^2)^3$

$y = (x^2 + 1)^{59}$

$y = \sqrt{1 + x^2}$

$y = (x + 1)^3(3 - x)^4$

$y = \sqrt[3]{\frac{x+1}{x-2}}$

$y = \frac{-2}{(4x-3)^3}$

The Chain Rule:

Alternative Form of the Chain Rule:

Example: Let $y = \sqrt{u}$ and $u = 1 - x^2$. Find $\frac{dy}{dx}$.

Differentiate the following:

$y = sin(x^2 + 1)$

$y = \text{sin}\pi x$

$y = \text{sec}x^2$

$y = \text{sin}^2 3x$

$y = \text{sin}(\text{cos}x)$

$y = \text{sin}(\text{cos}^2 x)$

Theorem: Derivative of the Exponential Function with base e

Let u be a differentiable function of x.

1. $\frac{d}{dx}[e^x] = e^x$

2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx} = e^u u'$

Find the derivative of the following:

$\frac{d}{dx}[e^{x^2-3x+1}]$

$\frac{d}{dx}[2e^{\frac{1}{x}}]$

$\frac{d}{dx}[\frac{1}{e^x}]$

$\frac{d}{dx}[e^{e^x}]$

Theorem: Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[lnx] = \frac{1}{x},\ x > 0$

2. $\frac{d}{dx}[lnu] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ u > 0$

Proof:

$\frac{d}{dx}[ln5x] =$

Differentiate $f(x) = ln\sqrt{2x-1}$

$\frac{d}{dx}(ln\frac{2x(x+1)^3}{\sqrt[3]{2x^2+1}}) =$

$\frac{d}{dx}[ln(lnx)] =$

The exponential function to the base a:

The logarithmic function to the base a:

Derivatives for Bases Other than e:

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\frac{d}{dx}[a^x] = (ln a)a^x$
2. $\frac{d}{dx}[a^u] = (ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[log_a x] = \frac{1}{(ln a)x}$
4. $\frac{d}{dx}[log_a u] = \frac{1}{(ln a)u}\frac{du}{dx}$

Proof:

Differentiate the following functions:

$y = 3^x$

$y = log_5 x^2$

$y = 2^{e^x}$

$y = log_3(cos 2x)$

Challenge question

Find the derivative of:

$$\ln\sqrt{\ln(\ln(\cos(x^3 - 2x)))}$$