Proving Trig Identities Video Lecture

Section 7.4

Course Learning Objectives:

1) Demonstrate an understanding of trigonometric functions and their applications.
2) Verify identities.

Weekly Learning Objectives:

1) Use algebra to simplify trigonometric expressions.
2) Use trigonometric formulas to establish and prove identities.

Proving Trigonometric Identities

Trigonometric identities allow us to rewrite trigonometric functions in terms of other trigonometric functions and as result allow us to simplify expressions and to make it easier to solve many trigonometric equations.

Example: To solve the equation for x: $\sin x \cot x = \frac{1}{2}$. It would simplify the equation considerably if we note that, because $\cot x = \dfrac{\cos x}{\sin x}$, the equation can be rewritten as $\cos x = \frac{1}{2}$. We know that if $x = \dfrac{\pi}{3}$ and $\dfrac{5\pi}{3}$, the equation is true.

Example: The expression $(1+\sin x)(\sec x - \tan x)$ can be rewritten in the following way.

$$\begin{aligned}(1+\sin x)(\sec x - \tan x) &= (1+\sin x)\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\ &= (1+\sin x)\left(\frac{1-\sin x}{\cos x}\right) \\ &= \frac{1-\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} \\ &= \cos x\end{aligned}$$

Clearly it would be easier to work with $\cos x$ than the expression $(1+\sin x)(\sec x - \tan x)$.

In this section we learn how to develop additional identities from the fundamental identities. Some of these additional identities are also worth memorizing because they are used so frequently. Recall that an identity is an equation that is true for ALL values of the variable for which the equation is defined. An equation that is true for only some values of the variable is called a conditional equation.

Identities	Conditional equations
$x^2 - 4 = (x-2)(x+2)$	$x^2 - 4 = 3x$
$\sin^2 x + \cos^2 x = 1$	$\sin x = \cos x$
$\tan x = \dfrac{\sin x}{\cos x}$	$\tan x = \cot x$

It is easy to show that an equation is not an identity. All we need to do is to find one value of the variable for which the equation is not true. In the conditional equations listed above, the first equation can be shown to not be an identity if we let $x = 1$, the second can be shown not to be an identity if we let $x = 0$. The third can be shown not to be an identity if we let $x = \dfrac{\pi}{3}$.

It is more difficult to prove that an equation is an identity because we need to show that the equation is true for all values of the variable. Clearly we cannot substitute all possible values and demonstrate that the equation is true for all of them. Thus we need an alternate procedure.

Basic Trigonometric Identities -
MEMORIZE and KNOW WELL!!!

Quotient Identities:

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Reciprocal Identities:

$\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

Pythagorean Identities:

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$

Or variations:

$\sin^2\theta = 1 - \cos^2\theta$ $\tan^2\theta = \sec^2\theta - 1$ $\cot^2\theta = \csc^2\theta - 1$
$\cos^2\theta = 1 - \sin^2\theta$

Even-Odd Identities:

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$ $\sec(-\theta) = \sec\theta$ $\cot(-\theta) = -\cot\theta$

The suggested procedure to follow is to:

1. Select one side of the equation and write it down. Your goal is to transform it to the other side of the equation. It is usually easier to start with the more complicated side.

2. Use algebra and the previous identities that we have memorized to help with this transformation. Frequently finding common denominators, factoring, eliminating complex fractions, and writing sums or differences of quotients as single quotients will help in this process. Sometimes it is helpful to write all of the functions in terms of the sine and the cosine.

3. Always keep your goal in mind. As you manipulate the expression on one side of the equation, you must keep in mind the expression on the other side as well.

NOTE: It is not acceptable to do something to both sides of the equation in attempting to prove an identity because doing something to both sides may result in changing an expression that is not an identity into one that is.
For example: $\sin x \neq -\sin x$, however if we square both sides, the equation becomes $\sin^2 x = \sin^2 x$ which is an identity.

Examples: Prove that the following equations are identities.

1. $\csc\theta \tan\theta = \sec\theta$

2. $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

3. $9\sec^2 x - 5\tan^2 x = 5 + 4\sec^2 x$

4. $$\frac{\cos\alpha}{1+\sin\alpha} + \frac{1+\sin\alpha}{\cos\alpha} = 2\sec\alpha$$

5. $$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\sin\theta + 1}{\cos\theta}$$

6. $$\frac{\sec\theta}{1+\sec\theta} = \frac{1-\cos\theta}{\sin^2\theta}$$

7. $$\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$$

8. $\dfrac{1}{1-\sin\beta} = \sec^2\beta + \tan\beta\sec\beta$

9. $\tan^2 x - \cot^2 x = \sec^2 x - \csc^2 x$

10. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$

Simplify the expression $\dfrac{1}{\sqrt{4 + x^2}}$ by letting $x = 2\tan\theta$

Show that the following equation is not an identity

$\sin(x + y) = \sin x + \sin y$