Infinite Series

Consider the sequence $a_n = \left\{ \frac{1}{2^{n-1}} \right\}$ i.e. $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \ldots$

Note: $\lim_{n\to\infty} a_n = 0$

Add the terms of the sequence:

Model by considering a string 2 feet long - cut in half, each piece is 1 foot long, cut one in half, ...

Analyze by forming a new sequence - called the sequence of partial sums.

S$_1 =$
S$_2 =$
S$_3 =$
S$_4 =$
S$_5 =$

The sequence of partial sums is . . .

Find a formula by observation:

An **infinite series** is

An infinite series **converges** if the sequence of partial sums converge.

An infinite series **diverges** if the sequence of partial sums diverge.

1. Consider the infinite series $\sum_{n=1}^{\infty} \frac{2}{(3n+1)(3n-2)}$. We can show that $S_1 = \frac{2}{4}, S_2 = \frac{4}{7}, S_3 = \frac{6}{10}, S_4 = \frac{8}{13}, S_5 = \frac{10}{16} \ldots$ Find a formula for $\left\{S_n\right\}$ and find the sum of the series.

2. Find a formula for S_n and find the sum of the series given $\sum_{n=1}^{\infty}(-1)^{n-1}$

3. What if we are given a formula for $\{S_n\}$ and want to find a formula for $\{a_n\}$? For example if $\{S_n\} = \left\{\frac{2n}{3n+1}\right\}$ i.e. $\frac{2}{4}, \frac{4}{7}, \frac{6}{10}, \frac{8}{13}, \frac{10}{16}, \ldots$
What is $a_1,\ a_2,\ \ldots a_{n-1}, a_n, a_{n+1} \ldots$? Find a formula for a_n

A **geometric series** is formed by adding the terms of a geometric sequence.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^{n-1} + ar^n + ar^{n+1} + \ldots$$

What is the sum of the geometric series?

$r = 1$

$r = -1$

$r \neq \pm 1$

Examples: Determine whether each of the following series converges or diverges. If the series converges, find its sum.

4. $\sum_{n=0}^{\infty} 2^n$

5. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

6. $\sum_{n=2}^{\infty} \left(\frac{4}{3}\right)^n$

7. $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$

8. $3 + \frac{3}{-4} + \frac{3}{(-4)^2} + \ldots + \frac{3}{(-4)^n} + \ldots$

9. $1+\frac{e}{3}+\frac{e^2}{3^2}+\ldots+\left(\frac{e}{3}\right)^{n-1}+\ldots$

10. $\sum_{n=0}^{\infty}(-5)^{n-1}4^{-n}$

11. Find all x such that

$$3+(x-1)+\frac{(x-1)^2}{3}+\frac{(x-1)^3}{3^2}+\ldots+\frac{(x-1)^{n-1}}{3^{n-2}}+\ldots$$

converges.

Theorem: Properties of Infinite Series

If $\sum a_n = A$, $\sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$

2. $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$

Theorem: Limit of nth term of a convergent series

If the series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence $\left\{ a_n \right\}$ converges to 0.

Theorem: n^{th}- term test for divergence:

If the sequence $\left\{ a_n \right\}$ does not converge to 0, then the series $\sum a_n$ diverges.

Use the nth term test to determine if the following series' diverge.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n+4}$ **13.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ **14.** $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Summary

<u>Tests for Convergence</u>

1) Find a formula for S_n

Converges if $\lim_{n\to\infty} S_n$ converges
and the Sum $= \lim_{n\to\infty} S_n$

2) Geometric Series

Converges if $0 < |r| < 1$
and the Sum $= \frac{a}{1-r}$

<u>Tests for Divergence</u>

1) nth term Test

Diverges if $\lim_{n\to\infty} a_n \neq 0$

2) Geometric Series

Diverges if $|r| > 1$

NOTE: If $\lim_{n\to\infty} a_n = 0$, can't conclude convergence or divergence.

Miscellaneous examples:

15. Change $.\overline{23}$ to a fraction

16. Find the sum of $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$. (Recall that to find the sum of a series the series must be geometric or we need to find a formula for S_n.
(This is called a **Telescoping Series**.)

17. Find the first term in the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ which is less than .0001. Does the series converge? If so, what is the sum?

18. Find the first term in the series $\sum_{n=1}^{\infty} (.01)^n$ which is less than .0001. Does the series converge? If so, what is the sum?

19. Find the first term in the series $\sum_{n=1}^{\infty} \frac{1}{n}$ which is less than .0001. Does the series converge? If so, what is the sum?

20. The radius of a large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process continues infinitely. Show that the surface area is infinite.