

# **Exponential Functions Video Lecture**

## **Section 5.3**

### **Course Learning Objectives:**

- 1) Graph exponential functions and use such graphs to solve applied problems and to understand the significance of attributes of the graph to such applied problems.**
- 2) Solve appropriate applications of determining compound interest and exponential growth and decay.**
- 3) Identify and articulate the significance of graphical components in a mathematical model/application.**

### **Weekly Learning Objectives:**

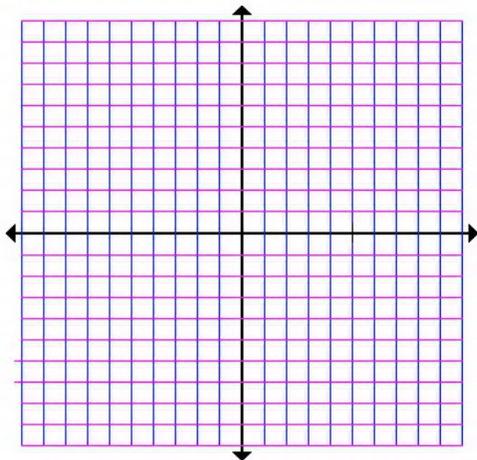
- 1) Evaluate exponential functions.**
- 2) Graph exponential functions with transformations.**
- 3) Define the number  $e$ .**
- 4) Apply the compound interest formula to applications.**
- 5) Determine the present value of a lump sum of money.**
- 6) Apply the compound continuously formula to applications.**
- 7) Solve exponential application problems including population and logistic growth.**

# Exponential Functions

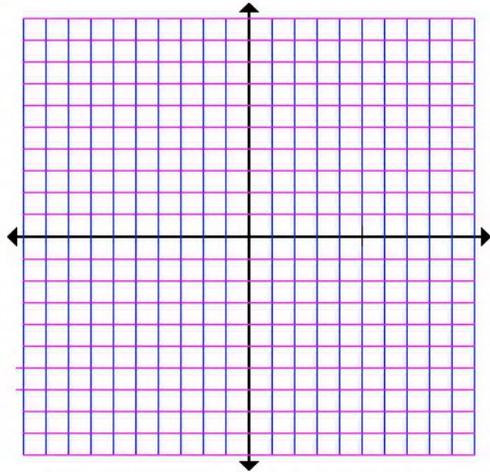
We will begin to study two of the most important functions in mathematics - the exponential function and its inverse, the logarithmic function. These functions can help us model a wide variety of natural phenomenon ranging from growth patterns in population or the value of money to radioactive decay, learning curves, spread of a virus or a rumor, carbon dating, and a host of other situations.

We will first look at the exponential function. The exponential function is a function of the form  $f(x) = a^x$ , where  $a > 0$  and  $x$  is a rational number.

1. Sketch the graph of  $f(x) = 2^x$



2. Sketch the graph of  $g(x) = \left(\frac{1}{2}\right)^x$

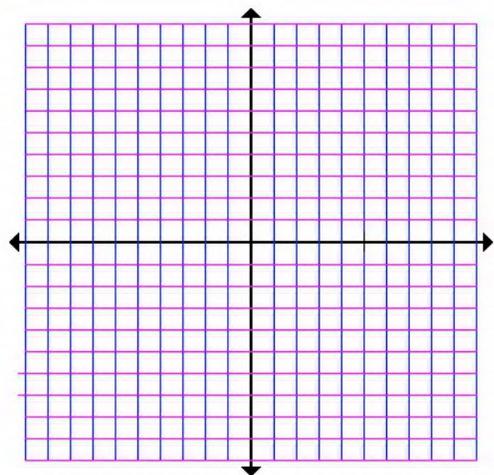


Characteristics of the graph of the exponential function  $f(x) = a^x$

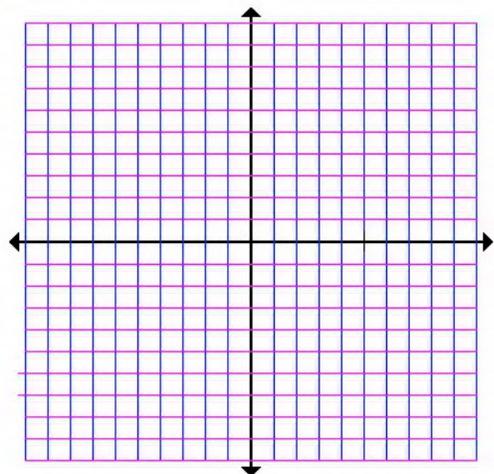
1. The graph of  $f(x) = a^x$  will always contain the point  $(0, 1)$ .
2. The graph will always contain the points  $(1, a)$  and  $\left(-1, \frac{1}{a}\right)$
3. When  $a > 1$  the graph is increasing. When  $0 < a < 1$  the graph is decreasing.
4. The graph is asymptotic to the negative  $x$ -axis when  $a > 1$ . ( $x \rightarrow -\infty, f(x) \rightarrow 0$ ). The graph is asymptotic to the positive  $x$ -axis when  $0 < a < 1$ . ( $x \rightarrow \infty, f(x) \rightarrow 0$ )
5. The domain is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ .
6. The function is one-to-one.
7. All previously discussed transformations apply to the graph of an exponential function.

Examples: Sketch the graph of each of the following indicating the transformations applied to the basic exponential function:

3.  $f(x) = 3^{x-1} + 2$



4.  $g(x) = -2^{2-x} - 3$



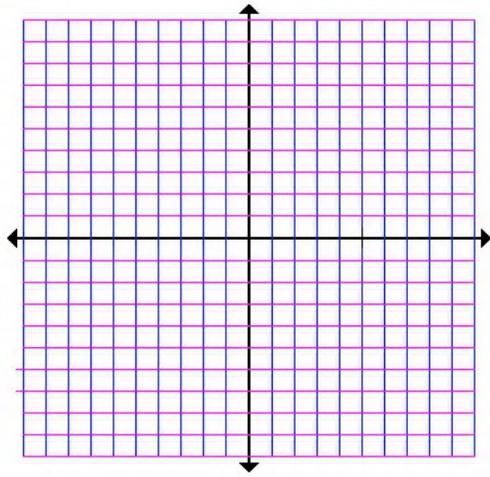
5. Find the equation of an exponential function of the form  $f(x) = ca^x$  that passes through the points  $(-1, 15)$  and  $(0, 5)$ .

We stated that any positive number can be used as the base in the exponential function. Bases 2 and 10 are convenient bases for many applications involving the exponential function but the most important base of the exponential function is the number  $e$ . The letter  $e$  is in recognition of a famous European mathematician of the 17<sup>th</sup> century Leonard Euler. The letter  $e$  is defined to be the value of the expression  $\left(1 + \frac{1}{n}\right)^n$  as  $n \rightarrow \infty$ .  $e$  is an irrational number.

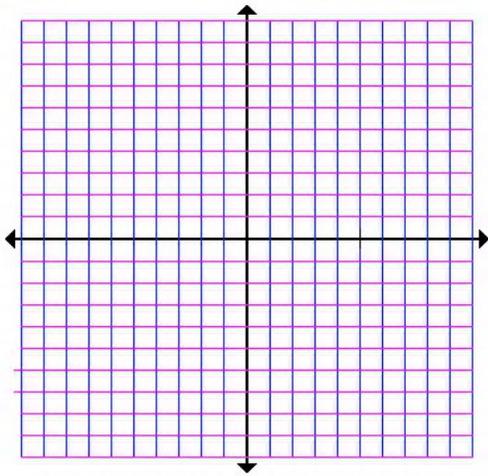
The natural exponential function is the exponential function  $f(x) = e^x$ . It is usually referred to as *the* exponential function. Note that the graph of  $f(x) = e^x$  lies between the graph of  $g(x) = 2^x$  and  $h(x) = 3^x$ .

Examples: Sketch the graph of each noting the transformations applied.

6.  $f(x) = e^{-x} + 1$



7.  $g(x) = e^{3-2x}$



Development of compound interest formula:

$P$  = Principal

$i$  = interest rate per time period

After 1 time period, Interest =  
Amount =

After 2 time periods, Interest =  
Amount =

After  $k$  time periods, Amount =

If the annual interest rate is  $r$  compounded  $n$  times per year, then in each time period the interest rate is  $i = \frac{r}{n}$  and there are  $n \cdot t$  time periods in  $t$  years.

So, compound interest is

Compound interest is calculated by the formula  $A = P\left(1 + \frac{r}{n}\right)^{nt}$

where

$A$  = amount after  $t$  years

$P$  = Principal

$r$  = interest rate per year

$n$  = number of times compounded per year

$t$  = number of years

**8.** Find the value of amount of an investment of \$10,000 invested for 5 years at 6% compounded quarterly.

**9.** Find the present value of \$10,000 invested at a rate of 8% compounded monthly for 5 years.

What happens as the number of compounding periods increase infinitely?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Continuously compounded interest is calculated by the formula  $A = Pe^{rt}$  where

- 10.** Find the amount if \$1,000 were invested at 11% compounded continuously for 1 year.

Population growth can be modeled with the exponential formula:

$$P(t) = P_0e^{rt}$$

where  $P_0$  is the initial population,  $r$  is the growth rate (as a decimal),  $t$  is time, and  $P(t)$  is the population at any time  $t$ .

This formula is similar to the continuously compounding formula.

- 11.** The population of a city was 250,000 and is growing at a relative rate of about 5% a year. What would you expect the population to be in 10 years?

A model which more accurately describes a population growth is a logistic growth model of the form  $f(t) = \frac{a}{b + ce^{-kt}}$ .

**12.** The population of a certain species of bird is limited by the type of habitat required for nesting. The population is modeled by the logistic growth formula  $f(t) = \frac{5600}{.5 + 27.5e^{-.044t}}$ .

a) What is the initial population?

b) What is the limiting value?

c) Find the graph.